Final Assessment | Spring - 2024

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Answer to the Question no- 1

(a)

Number Theory:

Number theory is a branch of mathematics dedicated to the study of integers and integer-valued functions. It deals with properties and relationships of numbers, particularly focusing on prime numbers, divisibility, and the solutions of equations in integers.

Example:

One of the most famous problems in number theory is **Fermat's Last Theorem**. The theorem states-

"There are no three positive integers a, b, and c that satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2."

This theorem, proposed by Pierre de Fermat in 1637, remained an open problem for more than 350 years. It was finally proven by Andrew Wiles in 1994.

(b)

Advantages of Generating Functions:

Generating functions are powerful tools in mathematics, particularly in combinatorics, number theory, and the study of sequences and series. Here are some of the key advantages of using generating functions:

1. **Simplification of Problems**:
   - Generating functions can transform complex combinatorial problems into problems about power series, which are often easier to handle.
2. **Solving Recurrence Relations**:
   - They provide a systematic way to solve linear recurrence relations. By converting the recurrence into an algebraic equation in terms of generating functions, solutions can be more readily found.
3. **Encoding Information Compactly**:
   - Generating functions encode an entire sequence of numbers into a single function. This can make it easier to manipulate and analyze the sequence.
4. **Facilitating Proofs**:
   - They can simplify combinatorial proofs, particularly those involving identities or sums, by transforming the problem into one of algebraic manipulation.
5. **Convolution and Combinatorial Interpretations**:
   - The product of generating functions corresponds to the convolution of sequences. This is particularly useful in counting problems where you need to combine different types of objects.
6. **Finding Explicit Formulas**:
   - They can be used to derive explicit formulas for terms of a sequence. Once a generating function is known, coefficients can often be extracted to give closed-form expressions.
7. **Handling Infinite Series**:
   - Generating functions are particularly useful for dealing with infinite series and understanding their properties.
8. **Applications in Various Fields**:
   - They find applications not only in combinatorics but also in probability theory, computer science (e.g., algorithm analysis), and physics (e.g., statistical mechanics).
9. **Transform Techniques**:
   - Generating functions can be manipulated using techniques such as differentiation, integration, and partial fractions, providing a wide array of methods to tackle various problems.

Answer to the Question no- 2

(a)

Predicate:

A predicate is a function that takes one or more inputs (often variables) and returns a Boolean value (true or false). It essentially asks a question about the input(s) and answers with a yes/no response.

Predicates are powerful tools because they allow us to express conditions and make decisions within algorithms and programs.

Here's an example of a predicate in mathematical analysis for computer science:

Predicate: isEven(x)
This predicate takes an integer x as input. It evaluates to true if x is even (divisible by 2) and false otherwise.

We can use this predicate in various ways:

- **Check if a number is even:**

```python
def isEven(x):
  return x % 2 == 0

number = 10
if isEven(number):
  print(number, "is even")
else:
  print(number, "is odd")
```

- **Filter a list of numbers:**

```python
numbers = [1, 4, 7, 9, 12]
even_numbers = list(filter(isEven, numbers))
print("Even numbers:", even_numbers)
```

In this example, the filter function iterates through the numbers list and keeps only the elements that satisfy the isEven predicate, resulting in a list of even numbers.

(b)

Here's a proof for the statement "If r is irrational, then $\sqrt{r}$ is also irrational" using proof by contrapositive:

1. State the contrapositive:

The contrapositive of the original statement is: "If $\sqrt{r}$ is rational, then r is rational." This means we will prove that if the square root of r is rational (can be expressed as a fraction), then r itself must also be rational.

2. Assume the opposite of the conclusion:

Assume for the sake of contradiction that $\sqrt{r}$ is rational. This means $\sqrt{r}$ can be expressed as a fraction in its simplest form, p/q, where p and q are integers with no common factors ($q \neq 0$).

3. Derive a contradiction:

Squaring both sides of the equation $\sqrt{r} = p/q$, we get:

$r = (\sqrt{r})^2 = (p/q)^2$

This simplifies to: $r = p^2/q^2$

Here, p^2 and q^2 are both integers (squares of integers), making r a rational number expressed as the fraction p^2/q^2.

4. Reach a contradiction:

This contradicts our initial assumption that r is irrational.

5. Conclusion:

Since assuming $\sqrt{r}$ to be rational leads to a contradiction, the original statement must be true. Therefore, if r is irrational, then $\sqrt{r}$ is also irrational.

Answer to the Question no- 3

Here's the proof that the standard deviation of a sequence of values $X_1 \ldots X_n$ is zero if all the values are equal to the mean:

1. **Definition of Standard Deviation:**

   Standard deviation (SD) is a measure of how spread out the data is from its average value (mean). It's calculated by finding the squared deviations from the mean, averaging them, and then taking the square root.

2. **Deviation from the Mean:**

   If all the values (X_1) are equal, let's call this common value "M" (the mean). In this case, the deviation of each value from the mean would be:

   $X_1 - M = M - M = 0$

   This holds true for all values in the sequence ($X_2, X_3, ..., X_n$) because they are all equal to M.

3. **Squared Deviations:**

   Since the deviations from the mean are all zero, the squared deviations from the mean will also be zero for each value $(X_1 - M)^2 = 0^2$.

4. **Average Squared Deviations:**

   When calculating the standard deviation, we average the squared deviations from the mean. If all the squared deviations are zero, their average will also be zero.

5. **Standard Deviation:**

   Finally, the standard deviation is calculated by taking the square root of the average squared deviations. Since the average squared deviations are zero, the square root of zero will be zero.

   Therefore, the standard deviation of a sequence of values $X_1 \ldots X_n$ is zero if all the values are equal to the mean.

   In essence, if all the data points are clustered at a single point (the mean), there's no spread in the data, and hence the standard deviation is zero.

Answer to the Question no- 4

Prime Numbers

A prime number is a natural number greater than 1 that has no positive divisors other than 1 and itself. For example, the number 7 is a prime because its only divisors are 1 and 7.

Prime numbers are fundamental in number theory due to their role as the "building blocks" of integers, given that every integer greater than 1 can be uniquely factored into prime numbers (this is known as the Fundamental Theorem of Arithmetic).

Real Numbers:
A real number is any number that can be found on the number line. This includes both rational numbers (such as integers and fractions) and irrational numbers.

Example of a Real Number:
5
$\sqrt{2}$
-3.14

Imaginary Numbers:
An imaginary number is a number that can be written as a real number multiplied by the imaginary unit i, where i is defined by the property $i^2 = -1$.

Example of an Imaginary Number:
$4i$
$-2i$
$\sqrt{7}.i$

Complex Numbers:
A complex number is a number that has both a real part and an imaginary part. It is generally written in the form $a+bi$, where a and b are real numbers and i is the imaginary unit.

Example of a Complex Number:
$3+2i$
$-1+4i$
$\sqrt{2}-i$

In summary:

- Real Number: 5
- Imaginary Number: $4i$
- Complex Number: $3+2i$

These examples illustrate the different types of numbers in the context of real and complex number systems.